Difference Equations II

96. Solve each of the following difference equations.

(a) $x_{n+2} - 3x_{n+1} + 2x_n = 0$, $x_0 = 5$, $x_1 = 7$.

(b) $x_{n+2} + 5x_{n+1} + 6x_n = 0$, $x_0 = 1$, $x_1 = 1$.

(c) $2x_{n+5} - 5x_{n+4} - 3x_{n+3} = 0$, $x_0 = 3$, $x_1 = -10$.

(d) $x_{n+2} + 81x_{n+1} - 17x_n = 0$, $x_0 = 0$, $x_1 = 0$.

(e) $x_n - 6x_{n-1} + 9x_{n-2} = 0$, $x_0 = 8$, $x_1 = 4$.

(f) $x_{n+2} - x_{n+1} - 6x_n = 0$, $x_0 = -1$, $x_1 = 1$.

Solution.

In each case we first solve the auxiliary equation and then use the initial conditions to evaluate any unknown constants that crop up.

(a) The auxiliary equation is

$$a^2 - 3a + 2 = 0$$

which has roots of $a = 1$ and $a = 2$. Thus the solution to this difference equation is

$$x_n = A \times 1^n + B \times 2^n = A + B2^n.$$

If we look at the initial conditions.

$$5 = x_0 = A + B2^0 = A + B$$
$$7 = x_1 = A + B2^1 = A + 2B$$

It is easy to see, this time, that $B = 2$ and hence $A = 3$ and the solution to the difference equations is

$$x_n = 3 + 2 \times 2^n = 3 + 2^{n+1}.$$

(b) The auxiliary equation is $a^2 + 5x + 6 = 0$ which has roots $a = -2$ and $a = -3$. Thus the solution to the difference equations is

$$x_n = A(-2)^n + B(-3)^n.$$

The initial conditions tell us the following.

$$1 = x_0 = A + B$$
$$1 = x_1 = -2A - 3B$$

This system of linear equations is only slightly harder to solve to show that $A = 4$ and $B = -3$. Thus the solution to the difference equation is

$$x_n = 4(-2)^n - 3(-3)^n = 4(-2)^n + (-3)^{n+1}.$$

(Note that in this part, like the previous part, you don't have to notice that $-3(-3)^n = (-3)^{n+1}$ to get the right answer.)

(c) The auxiliary equation is

$$2a^2 - 5a - 3 = 0$$

which has solutions $a = -1/2$ and $a = 3$. Thus the solution to the difference equation is

$$x_n = A\left(-\frac{1}{2}\right)^n + B \times 3^n.$$

The initial conditions tell us that

$$3 = x_0 = A + B$$
$$-10 = x_1 = -\frac{A}{2} + 3B$$

These are a bit harder to solve but you should find that $A = 38/7$ and $B = -17/7$ and so

$$x_0 = \frac{38}{7}\left(-\frac{1}{2}\right)^n - \frac{17}{7}3^n.$$

(d) The number look really nasty here except that the initial conditions are both 0. If we rearrange the difference equation we see that

$$x_{n+2} = -81x_{n+1} + 17x_n$$

It is easy to see now that $x_2 = 0$, as $x_0 = x_1 = 0$ and that indeed all the x_n's are equal to 0. Thus the solution to this difference equation is $x_n = 0$.

(e) The auxiliary equation is

$$a^2 - 6a + 9 = 0$$

which has repeated roots $a = 3$ and $a = 3$, thus the solution to the difference equation is

$$x_n = A3^n + Bn3^n.$$

Looking at the initial conditions.

$$8 = x_0 = A$$
$$4 = x_1 = 3A + 3B$$

Thus $A = 8$ and $B = -20/3$ and the solution to the difference equations is

$$x_n = 8 \times 3^n - \frac{20}{3} n 3^n.$$

(f) The auxiliary equation is

$$a^2 - a - 6 = 0$$

which has solutions $a = -2$ and $a = 3$. Thus the solution to the difference equation is

$$x_n = A(-2)^n + B3^n.$$

The initial conditions tell us

$$-1 = x_0 = A + B$$
$$1 = x_1 = -2A + 3B$$

This system has solutions $A = -4/5$ and $B = -1/5$ and hence the difference equation has the following solution.

$$x_n = -\frac{4}{5}(-2)^n - \frac{1}{5}3^n.$$

100. Start revision for the exam.