

AUSTRALIAN CATHOLIC UNIVERSITY

Castle Hill

Semester III, 1994

MM202: Calculus of Several Variables

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TIME ALLOWED: 2 hours, including 10 minutes reading time

INSTRUCTIONS TO STUDENTS

There are 10 questions on 3 pages, plus one page with useful formulae. Complete, careful, correct answers to 8 questions will obtain full marks. The marks available for each question are indicated at the end of each question.

This examination is worth 40% of your final mark.

You must give reasons, if none are given then you will get no marks for that part of the question.

You may attempt any and all questions.

Calculators are permitted.

The question paper must be handed in with your answers.

1. Integrate the following using a suitable substitution.

(a) $\int (x^2 + 5x + 7)^{90} (2x + 5) dx$

(b) $\int \sec^2 x \tan^2 x dx$

(c) $\int_0^1 x\sqrt{x+2} dx$

(d) $\int_0^{2\pi} \cos x e^{\sin x} dx$

(10)

2. Integrate the following by parts.

(a) $\int x \cos x \, dx$

(b) $\int x^2 e^{-x} \, dx$

(c) $\int \arctan x \, dx$, using the hint: $\arctan x = 1 \times \arctan x$.

(20)

3. (a) Define the functions $\sinh x$, $\cosh x$ and $\operatorname{sech} x$.

(b) Find the derivatives of the above functions.

(c) Show that $\sinh x \cosh y = \frac{1}{2}(\sinh(x+y) + \sinh(x-y))$.

(10)

4. (a) What is a level curve? Illustrate your answer with the function $z = 1 + 2xy$.

(b) Find the partial derivative of $z = x^2 + 2y^2x$ with respect to x using first principles. (20)

5. (a) Find the equation of the tangent plane to the surface

$$z = \sqrt{14 - x^2 - y^2}$$

at the point $(1, 2, 3)$. Does this plane intersect the surface at any other point?

(b) Find an approximation to

$$\sqrt[4]{15} - \sin \frac{1}{2}$$

using differentials. (Remember that $\pi \approx 3.14$)

(25)

6. Find all four second partial derivatives for the following functions and hence verify the mixed derivatives theorem.

(a) $z = \sin x \cos(xy)$

(b) $f(x, y) = x^2 + y^3 + x^4 \sin y$

(20)

7. (a) Find $\frac{dy}{dx}$ at the point $(2, -1)$ given

$$x^2y - y^2x + 2 = 0.$$

(b) Write down the equation of the tangent to this curve at that point.

(c) If a function $y = f(x)$ has inverse $y = g(x)$ show that

$$f'(x) = \frac{1}{g'(y)}.$$

Note that this is a different function from the one referred to in the first two parts. (20)

8. (a) Write down the total derivative rule.

(b) Given that

$$z = \cos x + \sin y$$

find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial \theta}$ if we convert to polar coordinates. That is, if we let $x = r \cos \theta$ and $y = r \sin \theta$. (15)

9. The ideal gas law states that

$$PV = kT$$

where P is the pressure, V is the volume, T the temperature and k a constant. Show that

$$\frac{\partial P}{\partial V} \frac{\partial V}{\partial T} \frac{\partial T}{\partial P} = -1.$$

(20)

10. (a) There are 4 second partial derivatives of a function of two variables. How many second partial derivatives of the function $f(x_1, x_2, x_3)$ are there?
- (b) How many 3rd partial derivatives of a function of 2 variables are there?
- (c) How many n^{th} partial derivatives of a function of m variables are there?
- (d) How many of these n^{th} derivatives are distinct?

(40)

Some Indefinite Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c, \quad n \neq -1$$

$$\int \frac{1}{x} dx = \log x + c, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax} + c, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax + c, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax + c, \quad a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax + c, \quad a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax + c, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \arctan \frac{x}{a} + c, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \arcsin \frac{x}{a} + c, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \log \left(x + \sqrt{x^2 - a^2} \right) + c, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \log \left(x + \sqrt{x^2 + a^2} \right) + c$$