AUSTRALIAN CATHOLIC UNIVERSITY

Castle Hill

Semester VIII

MT210: Abstract Algebra

LECTURER: W N Franzsen

September 1993

TIME ALLOWED: 2 hours, including 10 minutes reading time

INSTRUCTIONS TO STUDENTS

*There are 10 questions on 3 pages, you may attempt any and all questions. Complete, careful, correct solutions to 8 questions will obtain full marks. The number of marks available for each is given at the end of the question.*

*Non-programmable calculators are permitted.*

**All** *reasons must be given, in none are offered then you will receive no marks for that part of the question.*

*You may not leave the room during the first 30 minutes of the examination nor during the last 10 minutes.*

*Hand in the question paper with your attempt.*

**1.** (a) What is a group?

(b) Give an example of a finite group, including a list of all of its elements.

(c) What is an abelian group? Is your group in part (b) abelian?

(10)

**2.** (a) Write down elements of the subgroup generated by 3 in the group $(\mathbf{Z}, +)$.

(b) Verify that the subgroup in part (a) is a subgroup by showing that the appropriate axioms are satisfied.

(10)

**3.** What is a multiplication table? How can it be used to show that a finite set with an operation is a group?

Discuss in some detail.

(10)

**4.** Given the group $(G, \times)$ show

(a) If $a, b \in G$ then $\left(b^{-1}ab\right)^{-1} = b^{-1}a^{-1}b$.

(b) If $b \in G$ and $H$ is a subgroup of $G$ then so is

$$b^{-1}Hb = \{\, b^{-1}hb \,|\, h \in H \,\}.$$

(15)

**5.** Which of the following are groups? (Give reasons.)

(a) $\mathbf{Q}$ under multiplication.

(b) $\{\, i, (12), (34), (12)(34) \,\}$ under multiplication.

(c) $\{\, i, (1235), (3215), (13)(25) \,\}$ under multiplication.

(d) $\mathbf{C}^*$ under addition.

(15)

**6.** Find the period of the given element in the following groups.

(a) $(1234)(567)$ in $S_8$.

(b) $\dfrac{-1+\sqrt{-3}}{2}$ in $(\mathbf{C}^*, \times)$.

(c) $0$ in $(\mathbf{Z}, +)$.

(d) $(1254)(567)$ in $S_{10}$.

(15)

**7.** Find all the subgroups of $D_6$. You may assume that the order of a subgroup divides the order of the group. Here is a list of the elements of $D_6$.

$$D_6 = \{\, i, (123456), (135)(246), (14)(25)(36), (153)(264), (165432), \\ (26)(35), (12)(36)(45), (13)(46), (14)(23)(56), (15)(24), (16)(25)(34)\,\}$$

(30)

**8.** (a) Carefully define the period of an element $g$ in a group $(G, \times)$.

(b) Suppose that $g$ had period 20, show that $g^4$ has period 5. (Be careful.)

(c) Show that $o(g^k)$ is a factor of $o(g)$ for any integer $k$.

(25)

**9.** 2-cycles are sometimes called *transpositions*. Note that

$$(12)(23) = (123).$$

(a) Write (1234) as a product of transpositions.

(b) Write (12345) as a product of transpositions.

(c) Show that $(123 \cdots n-2, n-1)(n-1, n) = (123 \cdots n)$.

(d) Hence, or otherwise show that every element of $S_n$ can be written as a product of transpositions.
(Hint; use part (c) to prove by induction that every $k$-cycle can be written as a product of transpositions and proceed from there.)

(30)

**10.** Let $(G, \times)$ be a group with 4 elements, say

$$G = \{\, e, a, b, c\,\}.$$

(Where $e$ is the identity we know must be there.) Show that either $G$ is cyclic, or has multiplication table:

| | $e$ | $a$ | $b$ | $c$ |
|---|---|---|---|---|
| $e$ | $e$ | $a$ | $b$ | $c$ |
| $a$ | $a$ | $e$ | $c$ | $b$ |
| $b$ | $b$ | $c$ | $e$ | $a$ |
| $c$ | $c$ | $b$ | $a$ | $e$ |

Hence deduce that $G$ must be abelian. (Again you may assume that the order of a subgroup divides the order of the group.)

(40)