AUSTRALIAN CATHOLIC UNIVERSITY

CASTLE HILL

Semester 1

Algebra I and Calculus II

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TIME ALLOWED: 2 hours, plus 10 minutes reading time

INSTRUCTIONS TO STUDENTS

There are 10 questions on 3 pages. You may attempt any and all questions. I expect that the correct answer to 6 questions would be a very good result.

The marks available for each question is indicated at the end of each one.

Non-programmable calculators may be used. Show all working, if none is given then you might get no marks for that part.

1. Find matrices which represent each of the following transformations of the plane.

(a) An expansion by 2 in the x-direction and 3 in the y-direction.

(b) A reflection about the line $y = -x$.

(c) A shear with shear axis $y = 2x$ and a shear factor of 3.

(15)

2. Solve these systems of linear equations.

(a)

$$\begin{aligned} x + 2y &= 3 \\ 3x - 7y &= 4 \end{aligned}$$

(b)

$$\begin{aligned} x + y + z &= -2 \\ 2x - y - 3z &= 3 \\ -x + y + 3z &= -1 \end{aligned}$$

(c)

$$\begin{aligned} 2x + y + z &= 4 \\ 2x - y + 3z &= 4 \\ 2x + 3y - z &= 4 \end{aligned}$$

(20)

3. Find all the eigen-values and corresponding eigen-vectors for the matrix

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}.$$

(30)

4. (a) Evaluate the limit of each of the following sequences.

(*i*) $\left(\dfrac{n^2-1}{2n^2+3n+1}\right)$ (*ii*) $\left(\dfrac{n+\frac{1}{n}}{n^2+\frac{1}{n^2}}\right)$

(*iii*) $\left(\dfrac{\log n}{n}\right)$ (*iv*) $\left(\cos(n\pi)\right)$

(b) When can you use l'Hopital's rule to evaluate the limit of a sequence?

(20)

5. (a) Which of these series are summable?

(*i*) $1-\dfrac{1}{2}+\dfrac{1}{4}-\dfrac{1}{8}+\cdots$

(*ii*) $\displaystyle\sum_{i=2}^{\infty}\frac{1}{\sqrt{n^4-1}}$

(*iii*) $\displaystyle\sum_{i=1}^{\infty}\frac{n!}{n^n}$

(b) What is the radius of convergence of

$$1+x+x^2+x^3+\cdots?$$

(15)

6. (a) Calculate the Taylor polynomial about $\pi/2$ of degree 4 for the function $f(x)=\sin 2x$.

(b) Using the above polynomial estimate the value of $\sin 2$.

(10)

7. Evaluate the following integrals.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}dx$ (b) $\displaystyle\int_0^{\infty} \frac{1}{x^2}dx$

(c) $\displaystyle\int_0^2 \frac{1}{x-1}dx$ (d) $\displaystyle\int_1^{\infty} xe^{-x^2}\,dx$

(15)

8. Evaluate the following integrals.

(a) $\int \frac{x^4+1}{x^2+2} dx$ (b) $\int \sqrt{x} \log x \, dx$

(c) $\int \frac{1}{x^2-4x+13} dx$ (d) $\int (\log x)^2 \, dx$

(20)

9. (a) Calculate the volume of the solid obtained by rotating the region R under $y = \sqrt{1-x^2}$ and above $y = 0$ about the x-axis.

(b) Find the distance along the curve $y = x^{3/2}$ between $x = 0$ and $x = 4$.

(15)

10. (a) What do we mean by the statement "$\sum_{n=0}^{\infty} a_n$ is a summable series"?

(b) A sequence properly diverges to $+\infty$ if for any number M there is an integer N such that, if $n > N$ then $a_n > M$. Show that if (a_n) properly diverges to $+\infty$ then

$$\lim_{n\to\infty} \frac{1}{a_n} = 0.$$

(c) If $\lim_{n\to\infty} a_n = 0$ is it true that $\left(\frac{1}{a_n}\right)$ properly diverges to $+\infty$? If the answer is yes prove this, if it is no then give an example of a sequence where $\frac{1}{a_n}$ does not properly diverge to $+\infty$.

(40)